Unbounded Perturbations of Boson Equilibrium States in Fock Space

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Based on the linear coupling term in the non-relativistic quantum theory of matter-photon interactions, the coupling of a finite dimensional quantum system (finitely many finite-level atoms) with the boson gas (radiation field) in thermal equilibrium by means of perturbation theoretical methods is calculated. For the perturbation term of the Dyson expansion the unbounded interaction operator is taken. By a detailed analysis of the perturbation integrals and series, it is possible to derive the trace-class property of $e^{-\beta H}$, where H is the corresponding hamiltonian of the interacting systems and β the inverse temperature.

Key words: Expectation values of unbounded field observables; Thermal equilibrium of matter-photon systems; Selfadjointness and lower-boundedness of matter-photon hamiltonians; Perturbation expansions of semigroups.

1. Introduction

In quantum optics for many approximative treatments of the radiation field interacting with matter, in the fundamental hamiltonian the $\mathbf{A}^2$-term may be suppressed. The arguments range from perturbation theory to selection rules to canonical transformations. Using the linear coupling term as the interaction, we study a system of finitely many finite-level atoms coupled to the quantized electromagnetic field in thermal equilibrium. We particularly ask in which cases it is possible to formulate the equilibrium situation in the Fock representation. As far as we know, rigorous results concerning this question are unknown in the literature. In the tensor product of a finite-dimensional Hilbert space $\mathscr{H}$ and the Fock space there is no possibility for collective phenomena, phase transitions and superradiance. The present work indicates those equilibrium situations in which no such phenomena can appear. The radiation field has to be quantized in a bounded cavity so that the lowest (eigen-)frequency is strictly positive. The above mentioned phenomena then may appear in the thermodynamic limit, where one or both, the lattice system of the atoms or the photon system, are taken in the infinite volume limit. E.g. in [1] the limit for the (two-level) atoms is worked out, but only with a finite number of fixed radiation modes (ultraviolet cutoff), whereas in [2] (and in [3] for the ground states) the limit for the photon part – the two-level atom has been kept fixed – may lead to coupling constants with a singular infrared behaviour (infrared divergence), which gives rise to a discussion of the chirality of molecules [4]. The present approach to the treatment of the equilibrium problem in Fock space (exactly, in the tensor product with $\mathscr{H}$) uses perturbation theoretical methods and leads to a Dyson expansion for the exponential of the hamiltonian times minus the half inverse temperature, which converges with respect to the very strong Hilbert-Schmidt norm. Because of the unboundedness of the perturbation operator (interaction), the expansion is also of great mathematical interest.

Let us consider an assembly of finitely many atoms interacting with the quantized electromagnetic field. We assume the field to be enclosed in a cavity Λ ($\Lambda \subset \mathbb{R}^3$, open and bounded) with a sufficiently smooth boundary. For the atoms we take the finite-level approximation (cf. also [3], [5]). Then the various approximations of the radiation field neglecting the $\mathbf{A}^2$-term (cf. [6], [7], etc.) are summarized in the formal hamiltonian

$$H = \underbrace{A \otimes \mathbb{1}_r}_{\text{finite-level atoms}} + \underbrace{\mathbb{1}_N \otimes \sum_{n=0}^{\infty} \hbar\, \omega_n\, a_n^* a_n}_{\text{radiation field}} \tag{1.1}$$

$$+ \underbrace{\sum_{k=1}^{m} \left\{ B_k \otimes \left(\sum_{n=0}^{\infty} \overline{\lambda_n^{(k)}}\, a_n \right) + B_k^* \otimes \left(\sum_{n=0}^{\infty} \lambda_n^{(k)}\, a_n^* \right) \right\}}_{\text{linear coupling term}},$$

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where $m = N^2$, $A = A^*$ and $B_1, \ldots, B_m$ are elements of $\mathrm{M}(N \times N)$, the complex $N \times N$-matrices ($N \in \mathbb{N}$). For the B_k one may think of a basis for $\mathrm{M}(N \times N)$. $\mathbb{1}_N$ is the unit in $\mathrm{M}(N \times N)$ and $\mathbb{1}_r$ the unit of the yet to be specified Hilbert space of the radiation field. The a_n and a_n^* denote the annihilation and creation operators of the modes $\mathbf{u}_n$ [1] and they obey the usual canonical commutation relations. The $\lambda_n^{(k)}$ represent the coupling constants between the atoms and the modes $\mathbf{u}_n$ of the radiation field. Let the frequencies ω_n, $n \in \mathbb{N}_0$, be arranged in increasing order and assume the boundary condition such that ω_0 is strictly positive, i.e.

$$0 < \omega_0 \leq \omega_1 \leq \omega_2 \leq \omega_3 \leq \ldots \quad (1.2)$$

If S_Λ denotes the square root of the Laplacian times $c\hbar$, then $S_\Lambda \mathbf{u}_n = \hbar\omega \mathbf{u}_n$ and S_Λ is a selfadjoint strictly-positive operator acting in the sub-Hilbert space $\mathscr{K}$ of $\mathrm{L}^2(\Lambda, \mathbb{C}^3)$ spanned by the modes $\{\mathbf{u}_n \mid n \in \mathbb{N}_0\}$. $\mathscr{K}$ commonly is called the one-photon Hilbert space or the one-photon testfunction space and S_Λ the one-photon hamiltonian.

A similar formal hamiltonian is obtained, if the electromagnetic field is quantized in the whole euclidean space $\mathbb{R}^3$.

So far we have ignored the task of specifying an appropriate Hilbert space concerning the radiation field. In virtue of the infinite number of degrees of freedom involved, this is not a priori a well-defined affair. Here, as mentioned above, we are exclusively interested in the thermal equilibrium of the interacting atom-radiation system. Even, if the field is quantized in the whole $\mathbb{R}^3$, the equilibrium states of the free radiation field cannot be formulated in Fock space (cf. e.g. [8] and Section 3), so in this case there is no possibility of formulating the equilibrium situation of the interacting system in the tensor product of $\mathbb{C}^N$ and the Fock space over $\{\mathbf{f} \in \mathrm{L}^2(\mathbb{R}^3, \mathbb{C}^3) \mid \nabla \cdot \mathbf{f} = 0\}$. In [9] the general equilibrium situation is investigated by means of perturbation theoretical methods in the GNS-representation of the non-interacting composite system [2]. And in [11] are discussed the equilibrium states of the spin-boson model ($N = 2$) with the radiation field in the whole one-dimensional euclidean space $\mathbb{R}$. In the general case, because it is not possible to give $\sum_{n=0}^{\infty} \hbar\omega_n a_n^* a_n =: H_r$ a rigorous meaning in the GNS-Hilbert space, one has to replace H_r by a (GNS-)renormalized version, which is obtained by considering the dynamical problem in the Heisenberg picture. The present work is devoted to the question under which circumstances the equilibrium situation associated with (1.1) can be formulated in Fock space. It is well known that the quantization of the free electromagnetic field in a bounded cavity Λ yields

$$\mathrm{tr}(\mathrm{e}^{-\beta S_\Lambda}) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta\hbar\omega_n} < \infty \quad \text{for each inverse temperature } \beta > 0, \quad (1.3)$$

which together with (1.2) is equivalent to $\mathrm{e}^{-\beta H_r}$ being a well-defined positive trace-class operator on Fock space (cf. [12, Proposition 5.2.27] and Section 3). We remark that H_r is the second quantization of S_Λ (cf. [12, p. 8] and Section 2). Hence the equilibrium states of the quantized free radiation field enclosed in Λ are given in the Fock representation by the density operators $\mathrm{e}^{-\beta H_r}/\mathrm{tr}(\mathrm{e}^{-\beta H_r})$. Because of this reason we restrict our quantization of the field to a bounded region Λ of the euclidean space $\mathbb{R}^3$. Realistic estimations (see also [3] § 8, § 12 and Appendix IV, where the coupling constants for the 1s/2p- and the 2s/2p-states of hydrogen-like systems are calculated) show that $\sum_{n=0}^{\infty} |\lambda_n^{(k)}|^2 < \infty$, and hence with $a_n \equiv a(\mathbf{u}_n)$ we get well-defined smeared annihilation and creation operators, $a(\lambda^{(k)}) = \sum_{n=0}^{\infty} \overline{\lambda_n^{(k)}}\, a_n$ and $a^*(\lambda^{(k)}) = \sum_{n=0}^{\infty} \lambda_n^{(k)} a_n^*$, respectively, where $\lambda^{(k)} := \sum_{n=0}^{\infty} \lambda_n^{(k)} \mathbf{u}_n$, acting in the Fock space $\mathscr{F}_+(\mathscr{K})$ over the one-photon Hilbert space $\mathscr{K}$. Now (1.1) is a well-defined lower-bounded selfadjoint operator (for a proof, see Lemma 4.1 below) in the tensor product Hilbert space $\mathbb{C}^N \otimes \mathscr{F}_+(\mathscr{K})$ and writes

[1] The modes $\mathbf{u}_n$ are eigenfunctions of the Laplacian in Λ satisfying some boundary condition, $-\Delta\mathbf{u}_n = (\omega_n/c)^2 \mathbf{u}_n$ (c, light velocity), which are orthonormalized with respect to the scalar product of $\mathrm{L}^2(\Lambda, \mathbb{C}^3)$ and for which $\nabla \cdot \mathbf{u}_n = 0$ in Λ (Coulomb gauge). Because for the radiation field we take a very arbitrary cavity Λ and not only a simple box, it is not possible to distinguish specific directions of polarization, and hence the polarization is involved in the vector character of the modes $\mathbf{u}_n$.

[2] There are used operator algebraic methods and one starts with the C^*-algebra of the canonical commutation relations – the so-called Weyl algebra $\mathscr{W}$ (cf. Section 2) – over a infinite number of degrees of freedom (testfunction space), which represents the essential physical observables. The GNS-representation [10, Theorem 2.3.16] of $\mathscr{W}$ associated with the equilibrium state of the free electromagnetic field is not unitarily equivalent to the Fock representation of $\mathscr{W}$ and quasi-equivalent to the Fock representation, if and only if the equilibrium state is normal to the Fock representation, that is, can be defined by a density operator on the Fock space (cf. also Section 3).

shortly

$$H = A \otimes \mathbb{1}_r + \mathbb{1}_N \otimes H_r + \underbrace{\sum_{k=1}^{m} \{B_k \otimes a(\lambda^{(k)}) + B_k^* \otimes a^*(\lambda^{(k)})\}}_{=: P}. \tag{1.4}$$

It is not at all clear whether $e^{-\beta H}$ is of trace-class, and if it then can be used for the definition of the equilibrium state of the interacting atom-photon system. Now, the main result of the present paper is, that for arbitrary coupling constants $\lambda_n^{(k)}$ satisfying $\sum_{n=0}^{\infty} |\lambda_n^{(k)}|^2 < \infty$ for each $k \in \{1, \ldots, m\}$, or equivalently $\lambda^{(k)} \in \mathscr{K}$, the operator $e^{-\beta H}$ indeed is of trace-class on $\mathbb{C}^N \otimes \mathscr{F}_+(\mathscr{K})$. For the treatment of the problem we use a Dyson expansion for $e^{-\frac{\beta}{2}H}$ and regard the linear coupling term P as a perturbation. Despite the unboundedness of P, which arises from the unboundedness of the boson-annihilation and -creation operators, we show the convergence of the integrals and series in the perturbation expansion to be valid with respect to the Hilbert-Schmidt norm. This surprising result yields $e^{-\frac{\beta}{2}H}$ to be Hilbert-Schmidt, and consequently $e^{-\beta H}$ is trace-class in $\mathbb{C}^N \otimes \mathscr{F}_+(\mathscr{K})$. Now it is obvious to define the temperature states associated with the model hamiltonian H from (1.4) by the density operators $e^{-\beta H}/\mathrm{tr}(e^{-\beta H})$. These states agree with those obtained in [9] however in the GNS-representation. In some forthcoming publications (e.g. in the model of the Josephson junction weakly coupled to the microwave radiation) will be worked out the simultaneous infinite volume limits (number of the atoms $\to \infty$, cavity $\Lambda \to \mathbb{R}^3$), which allow some kind of singular infrared behaviour for the limit coupling constants. Also the investigation of KMS-properties of the equilibrium states is deferred to a subsequent work.

In detail we proceed as follows. In the next section we summarize some required facts about states on the boson C^*-Weyl algebra (describing the essential observables of the quantized electromagnetic field) and about Fock space. In the third section the expectation value functionals of the temperature states of the boson system (radiation field in the cavity Λ) are extended to unbounded field observables. As the central part of the present work, the selfadjointness and lower-boundedness of the interacting hamiltonian H and the Dyson expansion of $e^{-\frac{\beta}{2}H}$ are investigated in Section 4.

2. Representations of the CCR

In this section we establish our notation and also recall some required basic facts.

Let E be a complex pre-Hilbert space with inner product $\langle .,. \rangle$ (linear in the second factor) and $\mathscr{W}(E)$ the Weyl algebra over E, generated by Weyl operators $W(f)$, $f \in E$ ([12, p. 20]). In the GNS-representation $(\Pi_\omega, \mathscr{H}_\omega, \Omega_\omega)$ of a regular state ω on $\mathscr{W}(E)$, via Stone's theorem, the existence of selfadjoint field operators $\Phi_\omega(f)$ such that $\Pi_\omega(W(tf) = e^{it\Phi_\omega(f)}$ $\forall t \in \mathbb{R}$ is ensured. Using the $\Phi_\omega(f)$ one constructs the (smeared) annihilation and creation operators associated with ω

$$a_\omega(f) := \frac{1}{\sqrt{2}}(\Phi_\omega(f) + i\Phi_\omega(if)),$$

$$a_\omega^*(f) := \frac{1}{\sqrt{2}}(\Phi_\omega(f) - i\Phi_\omega(if)).$$

They are densely defined, closed, it is $a_\omega(f)^* = a_\omega^*(f)$, $f \mapsto a_\omega(f)$ is antilinear and $f \mapsto a_\omega^*(f)$ is linear and they fulfill the canonical commutation relations (CCR)

$$[a_\omega(f), a_\omega(g)] = [a_\omega^*(f), a_\omega^*(g)] = 0,$$

$$[a_\omega(f), a_\omega^*(g)] = \langle f, g \rangle \mathbb{1} \quad \forall f, g \in E$$

on a dense domain; for a proof see [12] Lemma 5.2.12. For analytic states ω the cyclic vector Ω_ω is in the domain of each polynomial of the field operators. This leads to a very natural extension of ω to polynomials of the unbounded field observables $\Phi_\omega(f)$, $f \in E$,

$$\omega(A\Phi_\omega(f_1) \ldots \Phi_\omega(f_n)) := \langle \Omega_\omega, \Pi_\omega(A)\, \Phi_\omega(f_1) \ldots \Phi_\omega(f_n)\, \Omega_\omega \rangle. \tag{2.1}$$

A special kind of entire analytic states are the gauge-invariant quasi-free states on $\mathscr{W}(E)$, which are determined by positive sesquilinear forms t on E with

$$\omega(W(f)) = \exp\{-\tfrac{1}{4}\|f\|^2 - \tfrac{1}{2}t(f,f)\} \quad \forall f \in E, \tag{2.2}$$

in which case we have $t(f, g) = \omega(a_\omega^*(g)\, a_\omega(f))$ $\forall f, g \in E$ (see e.g. [13]).

If $t \equiv 0$ we obtain in (2.2) the Fock state $\omega_{\mathscr{F}}$, whose GNS-representation is given by the Bose-Fock space $\mathscr{F}_+(\mathscr{K}) = \bigoplus_{n=0}^{\infty} \mathbb{P}_+(\otimes_n \mathscr{K})$, the Fock-Weyl operators $\Pi_{\mathscr{F}}(W(f)) = W_{\mathscr{F}}(f)$ and the vacuum vector $\Omega_{\mathscr{F}} = (1, 0, 0, \ldots)$. $\mathscr{K}$ is the completion of E, $\mathbb{P}_+$ denotes the symmetrisation operator and $\otimes_n \mathscr{K}$ the n-fold tensor product of $\mathscr{K}$ with itself. The (smeared) field, annihilation and creation operators in Fock space are denoted by $\Phi(f)$, $a(f)$ and $a^*(f)$. For the selfadjoint operator A on $\mathcal{K}$ we use Segal's notation $\mathrm{d}\Gamma(A)$ for its seond quantization (see [12, p. 8]). If A has purely discrete spectrum, then

$$\mathrm{d}\Gamma(A)=\sum_{n=0}^{\infty} \varepsilon_n a^*(e_n)\, a(e_n)$$

in the strong resolvent sense, where $Ae_n=\varepsilon_n e_n$, $n\in\mathbb{N}_0$. It is $N=\mathrm{d}\Gamma(\mathbb{1})$ for the number operator.

3. The Free Boson Field in Thermal Equilibrium

Let $\mathcal{K}$ be the one-particle Hilbert space and S the selfadjoint hamiltonian of a single boson (photon), satisfying the conditions

(I) $\exp\{-\beta S\}$ is trace-class on $\mathcal{K}$ for the inverse temperature $\beta>0$.

(II) $S>0$, that is $\langle\psi, S\psi\rangle>0$ $\forall\psi\in\mathcal{K}\setminus\{0\}$, and which by the spectral calculus is equivalent to $S\geq 0$ not having zero as its eigenvalue.

We remark, by (1.2) and (1.3) for each inverse temperature $\beta>0$ these two conditions are fulfilled for the selfadjoint operator S_Λ defined in the introduction. Obviously, for each $\beta>0$ one also can take $S:=-\Delta$ or $S:=\sqrt{-\Delta}$ for the Laplacian in an open bounded subset $\Lambda\subset\mathbb{R}^n$ $(n\geq 2)$ with some classical boundary conditions (the lowest eigenvalue has to be strictly larger than zero) and the choose $\mathcal{K}:=\mathrm{L}^2(\Lambda)$.

Condition (I) implies the existence of an orthonormal basis $(e_n)_{n\geq 0}$ for $\mathcal{K}$ consisting of eigenvectors of S. Let be $(\varepsilon_n)_{n\geq 0}$ the corresponding eigenvalues in increasing order, $Se_n=\varepsilon_n e_n$. Then $0<\varepsilon_0$. If $G:=\mathrm{d}\Gamma(S)$ and $G_\lambda:=\mathrm{d}\Gamma(S-\lambda\mathbb{1})=\mathrm{d}\Gamma(S)-\lambda N$ for $\lambda\in\mathbb{R}$, then $\exp\{-\beta G_\lambda\}$ is trace-class on $\mathcal{F}_+(\mathcal{K})$, if and only if $\lambda<\varepsilon_0$. Moreover

$$\operatorname{tr}(\mathrm{e}^{-\beta G_\lambda})=\sum_{m=0}^{\infty}(\mathrm{e}^{\beta\lambda})^m a_m<\infty \quad \forall\lambda\in\,]-\infty,\varepsilon_0[\,, \tag{3.1}$$

where

$$a_m:=\sum_{\substack{n_1,\ldots,n_m=0\\ n_1\leq\cdots\leq n_m}}^{\infty}\exp\{-\beta(\varepsilon_{n_1}+\cdots+\varepsilon_{n_m})\}=\operatorname{tr}_m(\mathrm{e}^{-\beta G}),$$

where tr_m denotes the trace on $\mathbb{P}_+(\otimes_m\mathcal{K})$. For a proof see e.g. [12] Proposition 5.2.27. With these assumptions the Gibbs canonical equilibrium state ω on $\mathcal{W}(\mathcal{K})$ is defined by

$$\omega(A):=\frac{\operatorname{tr}(\Pi_{\mathcal{F}}(A)\exp\{-\beta G\})}{\operatorname{tr}(\exp\{-\beta G\})} \qquad \forall A\in\mathcal{W}(\mathcal{K}). \tag{3.2}$$

As stated in [12] Proposition 5.2.28 our equilibrium state ω is quasi-free and gauge-invariant and given by (2.2) with the positive and bounded sesquilinear form

$$t(f,g)=\omega(a^*_\omega(g)\,a_\omega(f))$$
$$=\langle f,\mathrm{e}^{-\beta S}(\mathbb{1}-\mathrm{e}^{-\beta S})^{-1}g\rangle \quad \forall f,g\in\mathcal{K}.$$

In the sequel we use the notations $a^+(f):=a^*(f)$ and $a^-(f):=a(f)$ for the Fock creation and annihilation operators and $\|\,.\,\|_{\mathrm{HS}}$ for the Hilbert-Schmidt norm. Moreover, for $t\geq 0$ and $n\in\mathbb{N}$ we define

$$E_t^{(n)}:=\{\boldsymbol{t}=(t_1,\ldots,t_n)\in\mathbb{R}^n\,|\,0\leq t_1\leq\cdots\leq t_n\leq t\}\,. \tag{3.3}$$

The following lemma is essential for deriving the Hilbert-Schmidt properties stated in the introduction.

Lemma 3.1. *Let S be a selfadjoint operator in the Hilbert space $\mathcal{K}$ and $G:=\mathrm{d}\Gamma(S)$ its second quantization. Further let $\nu_k\in\{-,+\}$ and $f_k\in\mathcal{K}$ for each $k\in\{1,\ldots,n\}$, $n\in\mathbb{N}$. It follows:*

(i) *Assume $S\geq\varepsilon_0\mathbb{1}$ for some $\varepsilon_0>0$. Then for each $t\geq 0$ and $\boldsymbol{t}\in E_t^{(n)}$ the closure of the operator*

$$\mathrm{e}^{-t_1 G}a^{\nu_1}(f_1)\,\mathrm{e}^{-(t_2-t_1)G}a^{\nu_2}(f_2)\ldots$$
$$\ldots\mathrm{e}^{-(t_n-t_{n-1})G}a^{\nu_n}(f_n)\,\mathrm{e}^{-(t-t_n)G}$$

is defined on all of $\mathcal{F}_+(\mathcal{K})$ and bounded. The operator is closed if and only if $t_n<t$. Its closure we denote by $A^{t;n}_{\boldsymbol{\nu},\boldsymbol{f}}(\boldsymbol{t})$. We have

$$\|A^{t;n}_{\boldsymbol{\nu},\boldsymbol{f}}(\boldsymbol{t})\|\leq\left(\frac{\mathrm{e}^{2t\varepsilon_0}}{\sqrt{2t\varepsilon_0}}\right)^n\sqrt{n!}\,\|f_1\|\cdots\|f_n\|,$$

and the map $\boldsymbol{t}\in E_t^{(n)}\mapsto A^{t;n}_{\boldsymbol{\nu},\boldsymbol{f}}(\boldsymbol{t})\in\mathcal{L}(\mathcal{F}_+(\mathcal{K}))$ is continuous with respect to the operator norm.

(ii) *If S satisfies the conditions (I) and (II) for some $\beta>0$, then for each $\boldsymbol{t}\in E_{\beta/2}^{(n)}$ the operator $A^{\beta/2;n}_{\boldsymbol{\nu},\boldsymbol{f}}(\boldsymbol{t})$ is Hilbert-Schmidt on $\mathcal{F}_+(\mathcal{K})$. Moreover, there are constants $a,b>0$, independent on $n,\nu_1,\ldots,\nu_n$ and $f_1,\ldots,f_n$ (only dependent on S and β), such that*

$$\|A^{\beta/2;n}_{\boldsymbol{\nu},\boldsymbol{f}}(\boldsymbol{t})\|_{\mathrm{HS}}\leq a\,n!\left(\sum_{k=0}^{[n/2]}\frac{(2b)^{n-2k}}{(2k)!!\sqrt{(n-k)!}}\right)\|f_1\|\cdots\|f_n\| \quad \forall\boldsymbol{t}\in E_{\beta/2}^{(n)},$$

where $[n/2]:=\max\{m\in\mathbb{N}_0|m\leq n/2\}$ and $(2k)!!:=2\cdot 4\cdot 6\cdots 2k=2^k k!$. Further the map $\boldsymbol{t}\in E_{\beta/2}^{(n)}\mapsto A^{\beta/2;n}_{\boldsymbol{\nu},\boldsymbol{f}}(\boldsymbol{t})\in\mathrm{HS}(\mathcal{F}_+(\mathcal{K}))$ is continuous with respect to the Hilbert-Schmidt norm.

Proof: (i) If N is the number operator in the Fock space $\mathscr{F}_+(\mathscr{K})$, it is well know that (see e.g. [12, p. 19])

$$\| a^{\pm}(f)\psi \| \le \| f \| \, \| (N+\mathbb{1})^{\frac{1}{2}} \psi \| \quad \forall \psi \in \mathscr{D}(N^{\frac{1}{2}}),\ \forall f \in \mathscr{K}. \tag{3.4}$$

By the construction of the second quantization for each $\tau \ge 0$ the operator $e^{-\tau G}$ leaves the subspaces $\mathbb{P}_+(\otimes_m \mathscr{K})$ invariant and it is $\| e^{-\tau G}|_{\mathbb{P}_+(\otimes_m \mathscr{K})} \| \le \| e^{-\tau S} \|^m \le e^{-\tau \varepsilon_0 m}$. Now let $\phi = \bigoplus_{m=0}^{\infty} \phi_m \in \mathscr{F}_+(\mathscr{K})$. Then by orthogonality and (3.4)

$$\| e^{-t_1 G} a^{v_1}(f_1)\, e^{-(t_2-t_1)G} a^{v_2}(f_2) \dots e^{-(t_n-t_{n-1})G} a^{v_n}(f_n)\, e^{-(t-t_n)G} \phi \|^2$$

$$= \sum_{m=0}^{\infty} \| e^{-t_1 G} a^{v_1}(f_1)\, e^{-(t_2-t_1)G} a^{v_2}(f_2) \dots e^{-(t_n-t_{n-1})G} a^{v_n}(f_n)\, e^{-(t-t_n)G} \phi_m \|^2$$

$$\le \sum_{m=0}^{\infty} e^{-2t_1\varepsilon_0(m-n)} \| a^{v_1}(f_1)\, e^{-(t_2-t_1)G} a^{v_2}(f_2) \dots a^{v_n}(f_n)\, e^{-(t-t_n)G} \phi_m \|^2$$

$$\overset{(3.4)}{\le} \sum_{m=0}^{\infty} e^{-2t_1\varepsilon_0(m-n)} (m+n) \| f_1 \|^2 \| e^{-(t_2-t_1)G} a^{v_2}(f_2) \cdots e^{-(t_n-t_{n-1})G} a^{v_n}(f_n)\, e^{-(t-t_n)G} \phi_m \|^2$$

$$\overset{(3.4)}{\le} \sum_{m=0}^{\infty} e^{-2t_1\varepsilon_0(m-n)} (m+n) \| f_1 \|^2 e^{-2(t_2-t_1)\varepsilon_0(m-n)} (m+n-1) \| f_2 \|^2 \times \| e^{-(t_3-t_2)G} a^{v_3}(f_3) \dots e^{-(t_n-t_{n-1})G} a^{v_n}(f_n)\, e^{-(t-t_n)G} \phi_m \|^2$$

... and so on ...

$$\le \sum_{m=0}^{\infty} e^{-2t\varepsilon_0(m-n)} (m+n)(m+n-1)\dots(m+1) \| f_1 \|^2 \dots \| f_n \|^2 \| \phi_m \|^2$$

$$\overset{(*)}{\le} \left(\frac{e^{4t\varepsilon_0}}{2t\varepsilon_0} \right)^n n! \, \| f_1 \|^2 \dots \| f_n \|^2 \, \| \phi \|^2,$$

where in $(*)$ is used the inequality $\sup\{(m+1)(m+2)\dots(m+n)\, e^{-\alpha m} \mid m \in \mathbb{N}\} \le \left(\frac{e^{\alpha}}{\alpha} \right)^n n!$ for $\alpha > 0$. Therefore the closure of $e^{-t_1 G} a^{v_1}(f_1) \dots e^{-(t_n-t_{n-1})G} a^{v_n}(f_n)\, e^{-(t-t_n)G}$ is defined on all of $\mathscr{F}_+(\mathscr{K})$ and is a bounded operator. Using the domains of the annihilation and creation operators

$$\mathscr{D}(a^{\pm}(f)) = \left\{ \xi = \bigoplus_{n=0}^{\infty} \xi_n \in \mathscr{F}_+(\mathscr{K}) \,\middle|\, \sum_{n=0}^{\infty} \| a^{\pm}(f) \xi_n \|^2 < \infty \right\},$$

by analogous estimates one can easily check, that the operator is already closed for $t_n < t$. The stated continuity property follows analogously to the proof part (ii) (e) below.

(ii) For the proof of (ii) we need several steps.

(a) By (3.1) we have the holomorphy of the function $f\colon U \to \mathbb{C},\ z \mapsto \sum_{m=0}^{\infty} a_m z^m$, where $U := \{ z \in \mathbb{C} \mid |z| < e^{\beta\varepsilon_0} \}$. For a fixed $r \in]0, e^{\beta\varepsilon_0} - 1[$ let be $D_r := \{ z \in \mathbb{C} \mid |z-1| \le r \}$. Since D_r is compact, there exist some $a > 0$, such that $|f(\zeta)| \le a^2\ \forall \zeta \in D_r$. Mentioning $|\zeta| \le 1 + r\ \forall \zeta \in D_r$ and setting $b^2 := \frac{1+r}{r}$, it follows from Cauchy's inequalities (see e.g. [14, Prop. III.6.1])

$$\left| \left(\frac{\partial}{\partial z} \right)^n (z^n f(z)) \Big|_{z=1} \right| \le \frac{n!}{r^n} \max\{ |\zeta^n f(\zeta)| \mid \zeta \in D_r \} \le n!\, a^2 b^{2n}. \tag{3.5}$$

(b) For $v_1, \dots, v_l \in \{-, +\}$ and $g_1, \dots, g_l \in \mathscr{K}$, $l \in \mathbb{N}$, consider the operator

$$B_g^v := a^{v_1}(g_1) \dots a^{v_l}(g_l) \exp\left\{ -\frac{\beta}{2} G \right\}.$$

Obviously $B_g^{v*} B_g^v|_{\mathbb{P}_+(\otimes_m \mathscr{K})}$ maps $\mathbb{P}_+(\otimes_m \mathscr{K})$ into itself. Let be tr_m the trace, $\| \cdot \|_{\mathrm{tr}_m}$ the trace-norm and $\| \cdot \|_m$ the operator norm on $\mathbb{P}_+(\otimes_m \mathscr{K})$. Using the cyclicity of the trace, we calculate

$$\| B_g^{v*} B_g^v \|_{\mathrm{tr}_m} = \mathrm{tr}_m (B_g^{v*} B_g^v)$$

$$= \mathrm{tr}_m \left(e^{-\beta G} \left(\prod_{k=1}^{l} a^{v_k}(g_k) \right)^* \left(\prod_{k=1}^{l} a^{v_k}(g_k) \right) \right)$$

$$\le \left\| e^{-\beta G} \left(\prod_{k=1}^{l} a^{v_k}(g_k) \right)^* \left(\prod_{k=1}^{l} a^{v_k}(g_k) \right) \right\|_{\mathrm{tr}_m}$$

$$\le \| e^{-\beta G} \|_{\mathrm{tr}_m} \left\| \left(\prod_{k=1}^{l} a^{v_k}(g_k) \right)^* \left(\prod_{k=1}^{l} a^{v_k}(g_k) \right) \right\|_m$$

$$\overset{(3.4),(3.1)}{\le} \| g_1 \|^2 \dots \| g_l \|^2 (m+l) \dots (m+1)\, a_m .$$

Consequently $B_g^v|_{\mathbb{P}_+(\otimes_m \mathscr{K})}$ is Hilbert-Schmidt ([15], Section 6.2) from $\mathbb{P}_+(\otimes_m \mathscr{K})$ to $\mathbb{P}_+(\otimes_n \mathscr{K})$ with eventually $m \neq n$.

(c) In this step we prove the closure of $a^*(g_1)\dots a^*(g_l)\,\mathrm{e}^{-(\beta/2)G}a(g_{l+1})\dots a(g_n)$ to be Hilbert-Schmidt in $\mathscr{F}_+(\mathscr{K})$ and satisfying

$$\|a^*(g_1)\dots a^*(g_l)\,\mathrm{e}^{-(\beta/2)G}a(g_{l+1})\dots a(g_n)\|_{\mathrm{HS}} \leq a\,b^n\sqrt{n!}\,\|g_1\|\dots\|g_n\| \quad \forall g_1,\dots,g_n \in \mathscr{K} \quad \forall l \in \{0,1,,,,,n\}.$$

Remember, if $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}_3$ are Hilbert spaces and $T \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_2)$ is Hilbert-Schmidt and $S \in \mathscr{L}(\mathscr{K}_2, \mathscr{K}_3)$, then $ST \in \mathscr{L}(\mathscr{K}_1, \mathscr{K}_3)$ is Hilbert-Schmidt and $\|ST\|_{\mathrm{HS}} \leq \|S\|\,\|T\|_{\mathrm{HS}}$ and $\|T\|_{\mathrm{HS}} = \|T^*\|_{\mathrm{HS}}$, where $\mathscr{L}(\mathscr{K}_k, \mathscr{K}_l)$ denotes the bounded operators from $\mathscr{K}_k$ to $\mathscr{K}_l$, [15], Section 6.2.

Denote by $\|\cdot\|_{\mathrm{HS}_m}$ the Hilbert-Schmidt norm and by $\|\cdot\|_m$ the usual operator norm from $\mathbb{P}_+(\otimes_m \mathscr{K})$ to another Hilbert space. Because of $\mathscr{F}_+(\mathscr{K}) = \bigoplus_{m=0}^{\infty} \mathbb{P}_+(\otimes_m \mathscr{K})$ we get with the estimation of (b) and the holomorphic function $f(z)$ from (a)

Remark: In the same way we obviously obtain

$$\|B_g^v\|_{\mathrm{HS}}^2 = \mathrm{tr}(B_g^{v*} B_g^v) = \sum_{m=0}^{\infty} \|B_g^{v*} B_g^v\|_{\mathrm{tr}_m} \leq a^2 b^{2l}\, l!\, \|g_1\|^2 \dots \|g_l\|^2,$$

which leads to the inequality (3.7) below.

(d) Now let us consider the general operator $A_{v,f}^{\beta/2;n}(t)$. It is well known that for all $\tau \geq 0$ and all $g \in \mathscr{K}$ one has the relations

$$\begin{aligned} \mathrm{e}^{-\tau G} a^*(f) &= a^*(\mathrm{e}^{-\tau S} f)\,\mathrm{e}^{-\tau G}, \\ \mathrm{e}^{-\tau G} a(f) &= a(\mathrm{e}^{\tau S} f)\,\mathrm{e}^{-\tau G}. \end{aligned} \tag{3.6}$$

By systematic applications of (3.6) and the CCR it is possible to transform $A_{v,f}^{\beta/2;n}(t)$ into a linear combination of operators with normally ordered form

$$a^*(g_1)\dots a^*(g_k)\,\mathrm{e}^{-(\beta/2)G} a(h_1)\dots a(h_l),$$

where $k + l \leq n$ and $g_j = \mathrm{e}^{-t_p S} f_p$ for some $p \in \{1,\dots,n\}$ with $v_p = +$, and $h_j = \mathrm{e}^{-((\beta/2)-t_r)S} f_r$ for some $r \in \{1,\dots,n\}$

$$\begin{aligned}
&\|a^*(g_1)\dots a^*(g_l)\,\mathrm{e}^{-(\beta/2)G} a(g_{l+1})\dots a(g_n)\|_{\mathrm{HS}}^2 \\
&= \sum_{m=0}^{\infty} \|a^*(g_1)\dots a^*(g_l)\,\mathrm{e}^{-(\beta/2)G} a(g_{l+1})\dots a(g_n)|_{\mathbb{P}_+(\otimes_m \mathscr{K})}\|_{\mathrm{HS}}^2 \\
&\leq \sum_{m=0}^{\infty} \|a^*(g_1)\dots a^*(g_l)\|_{m-(n-l)}^2\, \|\mathrm{e}^{-(\beta/2)G} a(g_{l+1})\dots a(g_n)\|_{\mathrm{HS}_m}^2 \\
&= \sum_{m=0}^{\infty} \|a^*(g_1)\dots a^*(g_l)\|_{m-(n-l)}^2\, \|a^*(g_{l+1})\dots a^*(g_n)\,\mathrm{e}^{-(\beta/2)G}\|_{\mathrm{HS}_{m-(n-l)}}^2 \\
&\overset{(3.4),\,(b)}{\leq} \|g_1\|^2\dots\|g_n\|^2 \sum_{m=0}^{\infty} (m+n)\dots(m+1)\,a_m \\
&\overset{(a)}{=} \|g_1\|^2\dots\|g_n\|^2 \left[\left(\frac{\partial}{\partial z}\right)^n (z^n f(z))\Big|_{z=1}\right] \\
&\overset{(3.5)}{\leq} a^2 b^{2n}\, n!\, \|g_1\|^2\dots\|g_n\|^2 < \infty.
\end{aligned}$$

with $v_r = -$. This procedure we now demonstrate by an example:

$$\mathrm{e}^{-t_1 G} a^*(f_1)\, \mathrm{e}^{-(t_2-t_1)G} a(f_2)\, \mathrm{e}^{-(t_3-t_2)G} a(f_3)\, \mathrm{e}^{-(t_4-t_3)G} a^*(f_4)\, \mathrm{e}^{-\left(\frac{\beta}{2}-t_4\right)G}$$

$$\stackrel{(3.6)}{=} \mathrm{e}^{-t_1 G} a^*(f_1)\, \mathrm{e}^{-(t_4-t_1)G} a(\mathrm{e}^{-(t_4-t_2)S} f_2)\, a(\mathrm{e}^{-(t_4-t_3)S} f_3)\, a^*(f_4)\, \mathrm{e}^{-\left(\frac{\beta}{2}-t_4\right)G}$$

$$\stackrel{\mathrm{CCR}}{=} \mathrm{e}^{-t_1 G} a^*(f_1)\, \mathrm{e}^{-(t_4-t_1)G} a^*(f_4)\, a(\mathrm{e}^{-(t_4-t_2)S} f_2)\, a(\mathrm{e}^{-(t_4-t_3)S} f_3)\, \mathrm{e}^{-\left(\frac{\beta}{2}-t_4\right)G}$$

$$+ \langle \mathrm{e}^{-(t_4-t_3)S} f_3, f_4\rangle\, \mathrm{e}^{-t_1 G} a^*(f_1)\, \mathrm{e}^{-(t_4-t_1)G} a(\mathrm{e}^{-(t_4-t_2)S} f_2)\, \mathrm{e}^{-\left(\frac{\beta}{2}-t_4\right)G}$$

$$+ \langle \mathrm{e}^{-(t_4-t_2)S} f_2, f_4\rangle\, \mathrm{e}^{-t_1 G} a^*(f_1)\, \mathrm{e}^{-(t_4-t_1)G} a(\mathrm{e}^{-(t_4-t_3)S} f_3)\, \mathrm{e}^{-\left(\frac{\beta}{2}-t_4\right)G}$$

$$\stackrel{(3.6)}{=} a^*(\mathrm{e}^{-t_1 S} f_1)\, a^*(\mathrm{e}^{-t_4 S} f_4)\, \mathrm{e}^{-\frac{\beta}{2}G}\, a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-t_2\right)S} f_2\right) a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-t_3\right)S} f_3\right)$$

$$+ \langle \mathrm{e}^{-(t_4-t_3)S} f_3, f_4\rangle\, a^*(\mathrm{e}^{-t_1 S} f_1)\, \mathrm{e}^{-\frac{\beta}{2}G}\, a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-t_2\right)S} f_2\right)$$

$$+ \langle \mathrm{e}^{-(t_4-t_2)S} f_2, f_4\rangle\, a^*(\mathrm{e}^{-t_1 S} f_1)\, \mathrm{e}^{-\frac{\beta}{2}G}\, a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-t_3\right)S} f_3\right).$$

Clearly the same result is obtained by the formal calculation, for which we use the relation $\mathrm{e}^{-\tau G} a^{\pm}(g)\, \mathrm{e}^{\tau G} = a^{\pm}(\mathrm{e}^{\mp\tau S} g)$:

$$A^{\beta/2;\,n}_{\mathbf{v},\mathbf{f}}(\mathbf{t}) = \left(\prod_{k=1}^{n} a^{v_k}(\mathrm{e}^{-v_k t_k S} f_k)\right) \mathrm{e}^{-\frac{\beta}{2}G},$$

then use the CCR for normally ordering and finally with (3.6) bring $\mathrm{e}^{-\frac{\beta}{2}G}$ in the middle between the creation and annihilation operators.

Because of $\|\mathrm{e}^{-\tau S} g\| \le \|g\|\ \forall \tau \ge 0$ with (c) and the Cauchy-Schwarz inequality each summand of the normally ordered linear combination of $A^{\beta/2;\,n}_{\mathbf{v},\mathbf{f}}(\mathbf{t})$ is estimated in the Hilbert-Schmidt norm by $a b^N \sqrt{N!}\, \|f_1\| \ldots \|f_n\|$, where $N \in \{n, n-2, n-4, \ldots\}$ denotes the number of the creation plus the annihilation operators appearing in the summand.

But for an estimation of $A^{\beta/2;\,n}_{\mathbf{v},\mathbf{f}}(\mathbf{t})$ in the Hilbert-Schmidt norm we have to know how many summands have the same number N of creation plus annihilation operators. An answer gives the formula

$$(a(g)+a^*(g))^n = \sum_{k=0}^{[n/2]} \frac{n!}{(2k)!!\,(n-2k)!} \cdot \sum_{j=0}^{n-2k} \binom{n-2k}{j} \{\|g\|^{2k} a^*(g)^{n-2k-j} a(g)^j\},$$

which with use of the CCR is proven by induction [16]. Consequently, the number of the summands with the same number $N = n-2k$ of creation plus annihilation operators is less than

$$\frac{n!}{(2k)!!\,(n-2k)!} \sum_{j=0}^{n-2k} \binom{n-2k}{j} = \frac{n!}{(2k)!!\,(n-2k)!}\, 2^{n-2k}.$$

Finally we arrive at the stated estimation.

(e) Because of the decomposition of $A^{\beta/2;\,n}_{\mathbf{v},\mathbf{f}}(\mathbf{t})$ in a linear combination of operators with normally ordered form, it suffices to prove the continuity of the map

$$(t_1, \ldots, t_m) \in [0, \infty[^m \mapsto a^*(\mathrm{e}^{-t_1 S} g_1) \ldots a^*(\mathrm{e}^{-t_l S} g_l)\, \mathrm{e}^{-\frac{\beta}{2}G}\, a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-t_{l+1}\right)S} g_{l+1}\right) \ldots a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-t_m\right)S} g_m\right)$$

for each $g_1, \ldots, g_m \in \mathscr{K}$ and $m \in \mathbb{N}$. With the $\mathbb{R}$-linearity of $g \mapsto a^{\pm}(g)$ we obtain

$$\left\| \prod_{k=1}^{l} a^*(\mathrm{e}^{-\tau_k S} g_k)\, \mathrm{e}^{-\frac{\beta}{2}G} \prod_{k=l+1}^{m} a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-\tau_k\right)S} g_k\right) - \prod_{k=1}^{l} a^*(\mathrm{e}^{-t_k S} g_k)\, \mathrm{e}^{-\frac{\beta}{2}G} \prod_{k=l+1}^{m} a\left(\mathrm{e}^{-\left(\frac{\beta}{2}-t_k\right)S} g_k\right) \right\|_{\mathrm{HS}}$$

$$\leq \left\| \sum_{k=1}^{l} a^*(e^{-t_1 S} g_1) \dots a^*((e^{-t_k S} - e^{-\tau_k S}) g_k) \dots a^*(e^{-\tau_l S} g_l) \, e^{-\frac{\beta}{2} G} \left(\prod_{k=l+1}^{m} a\left(e^{-\left(\frac{\beta}{2}-\tau_k\right)S} g_k\right)\right) \right\|_{HS}$$

$$+ \left\| \left(\prod_{k=1}^{l} a^*(e^{-t_k S} g_k) \right) e^{-\frac{\beta}{2} G} \sum_{k=l+1}^{m} a\left(e^{-\left(\frac{\beta}{2}-t_{l+1}\right)S} g_{l+1}\right) \dots \right.$$

$$\left. \dots a\left(\left(e^{-\left(\frac{\beta}{2}-t_k\right)S} - e^{-\left(\frac{\beta}{2}-\tau_k\right)S}\right) g_k\right) \dots a\left(e^{-\left(\frac{\beta}{2}-\tau_m\right)S} g_m\right) \right\|_{HS}$$

$$\leq a b^m \sqrt{m!} \left\{ \sum_{k=1}^{l} \|g_1\| \dots \|g_{k-1}\| \, \|(e^{-t_k S} - e^{-\tau_k S}) g_k\| \, \|g_{k+1}\| \dots \|g_m\| \right\}$$

$$+ a b^m \sqrt{m!} \left\{ \sum_{k=l+1}^{m} \|g_1\| \dots \|g_{k-1}\| \left\| \left(e^{-\left(\frac{\beta}{2}-t_k\right)S} - e^{-\left(\frac{\beta}{2}-\tau_k\right)S}\right) g_k \right\| \, \|g_{k+1}\| \dots \|g_m\| \right\}$$

$\to 0$ for $\tau \to t$. □

As a side result of the above proof we have the estimation

$$\left\| a^{\nu_1}(f_1) \dots a^{\nu_n}(f_n) \exp\left\{-\frac{\beta}{2} G\right\} \right\|_{HS} \leq a b^n \sqrt{n!} \, \|f_1\| \dots \|f_n\| \tag{3.7}$$

for each $\nu_k \in \{-, +\}$ *and* $f_k \in \mathscr{K}$ and arbitrary $n \in \mathbb{N}$, from which one easily deduces two corollaries.

Corollary 3.2. *For each polynomial P of annihilation and creation operators and each $f \in \mathscr{K}$ it follows*

$$\sum_{m=0}^{\infty} \left(\frac{(it)^m}{m!} \Phi(f)^m P \exp\left\{-\frac{\beta}{2} G\right\} \right) = W_{\mathscr{F}}(tf) \, P \exp\left\{-\frac{\beta}{2} G\right\} \quad \forall t \in \mathbb{R}$$

and

$$\frac{\partial}{\partial t} W_{\mathscr{F}}(tf) \, P \exp\left\{-\frac{\beta}{2} G\right\} \Big|_{t=0} = i \Phi(f) \, P \exp\left\{-\frac{\beta}{2} G\right\}.$$

The convergence of the series and the differentiation is in the Hilbert-Schmidt norm.

Corollary 3.3. *It follows for the state ω defined by formula* (3.2)

$$\frac{\operatorname{tr}\left(\exp\left\{-\frac{\beta}{2} G\right\} \left(\prod_{k=1}^{M} a^{\nu_k}(f_k) \right)^* \Pi_{\mathscr{F}}(A) \left(\prod_{l=1}^{N} a^{\lambda_l}(g_l) \right) \exp\left\{-\frac{\beta}{2} G\right\} \right)}{\operatorname{tr}(\exp\{-\beta G\})}$$

$$= \left\langle \left(\prod_{k=1}^{M} a_{\omega}^{\nu_k}(f_k) \right) \Omega_{\omega}, \; \Pi_{\omega}(A) \left(\prod_{l=1}^{N} a_{\omega}^{\lambda_l}(g_l) \right) \Omega_{\omega} \right\rangle$$

for all $f_1, \dots, f_M, g_1, \dots, g_N \in \mathscr{K}$; $\nu_1, \dots, \nu_M, \lambda_1, \dots, \lambda_N \in \{-, +\}$; $M, N \in \mathbb{N}_0$ *and all* $A \in \mathscr{W}(\mathscr{K})$.

Using Lemma 3.1 we can extend the Gibbs state ω of (3.2) to polynomials of Fock annihilation and creation operators $a(g)$ and $a^*(g)$. If P is such a polynomial, we define the extension by

$$\omega(AP) := \frac{\operatorname{tr}\left(e^{-\frac{\beta}{2} G} \Pi_{\mathscr{F}}(A) \, P \, e^{-\frac{\beta}{2} G} \right)}{\operatorname{tr}(e^{-\beta G})}, \quad A \in \mathscr{W}(\mathscr{K}).$$

Because of Corollary 3.3 this extension agrees with the one defined by (2.1): If P is a polynomial in $a(f)$ and $a^*(f)$ and P_ω the same polynomial in $a_\omega(f)$ and $a_\omega^*(f)$, then $\omega(AP) = \omega(AP_\omega)$ $\forall A \in \mathscr{W}(\mathscr{K})$.

4. Perturbation Expansions

In this section we first treat the selfadjointness and the lower-boundedness of the perturbed hamiltonian from (1.4). Then we investigate the Hilbert-Schmidt properties stated in the introduction. By $\odot$ we denote the algebraic tensor product.

Lemma 4.1. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert spaces and $S>0$ a selfadjoint operator on $\mathscr{K}$ and $\mathrm{d}\Gamma(S)=:G$ the second quantization of S in $\mathscr{F}_+(\mathscr{K})$. Let A be a selfadjoint operator on $\mathscr{H}$ bounded from below, $B_1,\ldots,B_m \in\mathscr{L}(\mathscr{H})$ and $f_1,\ldots,f_m\in\mathscr{D}(S^{-\frac{1}{2}})$ for an arbitrary $m\in\mathbb{N}$. Furthermore let*

$$K:=A\otimes\mathbb{1}+\mathbb{1}\otimes G$$

and

$$P:=\sum_{k=1}^{m}(B_k\otimes a(f_k)+B_k^*\otimes a^*(f_k))$$

be operators on $\mathscr{H}\otimes\mathscr{F}_+(\mathscr{K})$. It follows: P is relatively bounded with respect to K with relative bound zero. Therefore by [17, Theorems V.4.4 and V.4.11] *the operator $H:=K+P$ is bounded from below, selfadjoint, has domain $\mathscr{D}(H)=\mathscr{D}(K)$ and each core of K is a core of H.*

Proof: By construction of the second quantization the subspace

$$D=\bigcup_{m=1}^{\infty}\bigoplus_{n=0}^{m}\mathbb{P}_+(\odot_n\mathscr{D}(S)) \tag{4.1}$$

is a core of G. Let $\alpha:=\|S^{-\frac{1}{2}}f\|$ for some $f\in\mathscr{D}(S^{-\frac{1}{2}})$ and $p:=|f\rangle\langle f|$. We have

$$\langle g,pg\rangle=|\langle f,g\rangle|^2=|\langle S^{-\frac{1}{2}}f,S^{\frac{1}{2}}g\rangle|^2\le\alpha^2\|S^{\frac{1}{2}}g\|^2$$
$$=\alpha^2\langle g,Sg\rangle\quad\forall g\in\mathscr{D}(S).$$

Consequently, $a^*(f)a(f)=\mathrm{d}\Gamma(p)\le\alpha^2\,\mathrm{d}\Gamma(S)=\alpha^2G$ on D and further $\mathbb{1}\otimes a^*(f)a(f)\le\alpha^2\mathbb{1}\otimes G$ on $\mathscr{H}\odot D$. Now let $\psi\in\mathscr{D}(A)\odot D$, $A\ge-\gamma\mathbb{1}$ for some $\gamma\ge0$ and $B\in\mathscr{L}(\mathscr{H})$. Using the CCR we calculate

$$\|(B\otimes a(f)+B^*\otimes a^*(f))\psi\|^2$$
$$\le\|B\|^2(\|\mathbb{1}\otimes a(f)\psi\|+\|\mathbb{1}\otimes a^*(f)\psi\|)^2$$
$$\le2\|B\|^2(\|\mathbb{1}\otimes a(f)\psi\|^2+\|\mathbb{1}\otimes a^*(f)\psi\|^2)$$
$$\overset{(\mathrm{CCR})}{\le}4\|B\|^2\langle\psi,\mathbb{1}\otimes a^*(f)a(f)\psi\rangle+2\|B\|^2\|f\|^2\|\psi\|^2$$
$$\le4\|B\|^2\alpha^2\langle\psi,\mathbb{1}\otimes G\psi\rangle+2\|B\|^2\|f\|^2\|\psi\|^2$$
$$=4\|B\|^2\alpha^2\langle\psi,(A\otimes\mathbb{1}+\mathbb{1}\otimes G)\psi\rangle$$
$$-4\|B\|^2\alpha^2\langle\psi,A\otimes\mathbb{1}\psi\rangle+2\|B\|^2\|f\|^2\|\psi\|^2$$
$$\le\left(\frac{a}{\sqrt{\varepsilon}}\|\psi\|\right)\sqrt{\varepsilon}\,\|(A\otimes\mathbb{1}+\mathbb{1}\otimes G)\psi\|+b\|\psi\|^2$$
$$\le\varepsilon\|(A\otimes\mathbb{1}+\mathbb{1}\otimes G)\psi\|^2+\left(\frac{a^2}{\varepsilon}+b\right)\|\psi\|^2\quad\forall\varepsilon>0,$$

where $a=4\|B\|^2\alpha^2$ and $b=2\|B\|^2(2\alpha^2\gamma+\|f\|^2)$. Consequently for each $\varepsilon>0$ there exist some $a_\varepsilon\in\mathbb{R}$ such that

$$\|(B_k\otimes a(f_k)+B_k^*\otimes a^*(f_k))\psi\|$$
$$\le\frac{\varepsilon}{m}\|(A\otimes\mathbb{1}+\mathbb{1}\otimes G)\psi\|+\frac{a_\varepsilon}{m}\|\psi\|.$$

From the triangle inequality we get $\|P\psi\|\le\varepsilon\|K\psi\|+a_\varepsilon\|\psi\|\quad\forall\psi\in\mathscr{D}(A)\odot D$. □

The proof of Lemma 4.1 is inspired by a similar proof due to Davies [18] for form perturbations.

Now we arrive at the main theorem of the present work. Remember the definition of the set $E_t^{(n)}$, which is given in (3.3).

Theorem 4.2. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert spaces. For $\varepsilon_0>0$ let be $S\ge\varepsilon_0\mathbb{1}$ a selfadjoint operator in $\mathscr{K}$ and $\mathrm{d}\Gamma(S)=:G$. Let further $A,B_1,\ldots,B_m\in\mathscr{L}(\mathscr{H})$, $A=A^*$ and $f_1,\ldots,f_m\in\mathscr{K}$ with an arbitrary $m\in\mathbb{N}$, and*

$$K:=A\otimes\mathbb{1}+\mathbb{1}\otimes G$$

and

$$P:=\sum_{k=1}^{m}(B_k\otimes a(f_k)+B_k^*\otimes a^*(f_k)).$$

It follows

(i) *For each $n\in\mathbb{N}$ and $\boldsymbol{t}\in E_t^{(n)}$, $t\ge0$, the closure of the operator*

$$\mathrm{e}^{-t_1K}P\,\mathrm{e}^{-(t_2-t_1)K}P\,\mathrm{e}^{-(t_3-t_2)K}P\ldots$$
$$\ldots\mathrm{e}^{-(t_n-t_{n-1})K}P\,\mathrm{e}^{-(t-t_n)K}$$

is defined on all of $\mathscr{H}\otimes\mathscr{F}_+(\mathscr{K})$ and is bounded. The operator is closed if and only if $t_n<t$. The closure will be denoted by $Q_t^{(n)}(\boldsymbol{t})$. The map $E_t^{(n)}\ni\boldsymbol{t}\mapsto Q_t^{(n)}(\boldsymbol{t})$ is continuous in the operator norm and with $H:=K+P$ one has

$$\mathrm{e}^{-tH}=\mathrm{e}^{-tK}+\sum_{n=1}^{\infty}(-1)^n\int_{E_t^{(n)}}Q_t^{(n)}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}, \tag{4.2}$$

where the integrals and the series converge with respect to the operator norm.

(ii) *Let $\mathscr{H}$ be finite dimensional. If for some $\beta>0$ the operator $\mathrm{e}^{-\beta S}$ is trace-class, then*

$$Q_{\beta/2}^{(n)}(\boldsymbol{t})\in\mathsf{HS}(\mathscr{H}\otimes\mathscr{F}_+(\mathscr{K}))\quad\forall\boldsymbol{t}\in E_{\beta/2}^{(n)}$$

and the map $E_{\beta/2}^{(n)}\ni\boldsymbol{t}\mapsto Q_{\beta/2}^{(n)}(\boldsymbol{t})$ is continuous in the Hilbert-Schmidt norm and for $t=\beta/2$ in (4.2) the integrals and the series converge with respect to *the Hilbert-Schmidt norm. Consequently* $e^{-\frac{\beta}{2}H}$ *is Hilbert-Schmidt.*

Remark: Because of $S \geq \varepsilon_0 \mathbb{1}$ the operator $S^{-\frac{1}{2}}$ is bounded and thus $\mathscr{D}(S^{-\frac{1}{2}}) = \mathscr{K}$. Consequently, by Lemma 4.1 H is selfadjoint and bounded from below.

Proof: For a better survey we first prove the theorem for $m=1$ and later on we consider the general case. Thus let be $P = B \otimes a(f) + B^* \otimes a^*(f)$ and $B^+ := B^*$, $B^- := B$.

(i) Let us write $v = (v_1, \ldots, v_n)$ with $v_k \in \{-, +\}$ and set $t_0 := 0$. We get by multiplying out

$$Q_t^{(n)}(\boldsymbol{t}) = \sum_v \left(\prod_{k=1}^n (e^{-(t_k - t_{k-1})A} B^{v_k}) e^{-(t-t_n)A} \right) \otimes \left(\prod_{k=1}^n (e^{-(t_k - t_{k-1})G} a^{v_k}(f)) e^{-(t-t_n)G} \right). \tag{4.3}$$

With Lemma 3.1 (i) and since there are 2^n summands in $\sum_v$, we get with $\| e^{-\tau A} \| \leq e^{\tau \|A\|}$ for $\tau \geq 0$

$$\| Q_t^{(n)}(\boldsymbol{t}) \| \leq \sum_v \left\| \prod_{k=1}^n (e^{-(t_k - t_{k-1})A} B^{v_k}) e^{-(t-t_n)A} \right\| \cdot \left\| \prod_{k=1}^n (e^{-(t_k - t_{k-1})G} a^{v_k}(f)) e^{-(t-t_n)G} \right\| \tag{4.4}$$
$$\leq e^{t\|A\|} (2 \|B\| \, \|f\| \, c_t)^n \sqrt{n!}$$

$$\text{with} \quad c_t := \frac{e^{2t\varepsilon_0}}{\sqrt{2t\varepsilon_0}}.$$

The continuity of $Q_t^{(n)}: E_t^{(n)} \to \mathscr{L}(\mathscr{H} \otimes \mathscr{F}_+(\mathscr{K}))$ in the operator norm immediately follows from (4.3) and Lemma 3.1 (i). Thus $U_t^{(n)} := (-1)^n \int_{E_t^{(n)}} d\boldsymbol{t}\, Q_t^{(n)}(\boldsymbol{t})$ exists as a Riemann integral converging in the operator norm. Using $\int_{E_t^{(n)}} d\boldsymbol{t} = \frac{t^n}{n!}$ we have

$$\| U_t^{(n)} \| \leq e^{t\|A\|} \frac{(2 \|B\| \, \|f\| \, t c_t)^n}{\sqrt{n!}}.$$

Consequently $\sum_{n=0}^{\infty} U_t^{(n)}$ converges in the operator norm. Let $V_t := \sum_{n=0}^{\infty} U_t^{(n)}$. The proof that $V_s V_t = V_{s+t}$ for all $s, t \geq 0$ is straightforward. Now proceed similar to [19, p. 69] to determine the generator of the semigroup $(V_t)_{t \in \mathbb{R}}$. For the latter use the fact that $\mathscr{H} \odot D$ is a core for K, where D is defined in (4.1), and Lemma 4.1.

(ii) Because $\mathscr{H}$ is finite dimensional, the Hilbert-Schmidt norm on $\mathscr{L}(\mathscr{H}) = \mathsf{HS}(\mathscr{H})$ is equivalent to the usual operator norm, that is, there exist some $d > 0$, such that $\|X\|_{\mathsf{HS}} \leq d \|X\|$ for all $X \in \mathscr{L}(\mathscr{H})$. By Lemma 3.1 (ii) we get with (4.3)

$$\| Q_{\beta/2}^{(n)}(\boldsymbol{t}) \|_{\mathsf{HS}(\mathscr{H} \otimes \mathscr{F}_+(\mathscr{K}))} \leq \sum_v \left\| \left(\prod_{k=1}^n e^{-(t_k - t_{k-1})A} B^{v_k} \right) e^{-\left(\frac{\beta}{2} - t_n\right)A} \right\|_{\mathsf{HS}(\mathscr{H})} \cdot \left\| \left(\prod_{k=1}^n e^{-(t_k - t_{k-1})G} a^{v_k}(f) \right) e^{-\left(\frac{\beta}{2} - t_n\right)G} \right\|_{\mathsf{HS}(\mathscr{F}_+(\mathscr{K}))} \tag{4.5}$$
$$\leq a d e^{\frac{\beta}{2}\|A\|} n! \, (2 \|B\| \, \|f\|)^n \left(\sum_{k=0}^{[n/2]} \frac{(2b)^{n-2k}}{(2k)!! \sqrt{(n-2k)!}} \right) \quad \forall \boldsymbol{t} \in E_{\beta/2}^{(n)} \quad \forall n \in \mathbb{N}.$$

The continuity of $Q_{\beta/2}^{(n)}: E_{\beta/2}^{(n)} \to \mathsf{HS}(\mathscr{H} \otimes \mathscr{F}_+(\mathscr{K}))$ with respect to the Hilbert-Schmidt norm follows from (4.3) and Lemma 3.1 (ii). Hence the Riemann integrals of $U_{\beta/2}^{(n)}$ converge in Hilbert-Schmidt norm and because of $\int_{E_{\beta/2}^{(n)}} d\boldsymbol{t} = \frac{(\beta/2)^n}{n!}$ we get

$$\| U_{\beta/2}^{(n)} \|_{\mathsf{HS}} \leq a d e^{\frac{\beta}{2}\|A\|} (\beta \|B\| \, \|f\|)^n \left(\sum_{k=0}^{[n/2]} \frac{(2b)^{n-2k}}{(2k)!! \sqrt{(n-2k)!}} \right) \quad \forall n \in \mathbb{N}.$$

Consequently, separating $\sum_{n=0}^{\infty} = \sum_{n,\,\text{even}} + \sum_{n,\,\text{odd}}$ we obtain

$$\sum_{n=0}^{\infty} \| U_{\beta/2}^{(n)} \|_{\mathsf{HS}} \leq a d e^{\frac{\beta}{2}\|A\|} \sum_{n=0}^{\infty} (\beta \|B\| \, \|f\|)^n \left(\sum_{k=0}^{[n/2]} \frac{(2b)^{n-2k}}{(2k)!! \sqrt{(n-2k)!}} \right)$$
$$= a d e^{\frac{\beta}{2}\|A\|} \left[\sum_{m=0}^{\infty} (\beta \|B\| \, \|f\|)^{2m} \left(\sum_{k=0}^{m} \frac{(2b)^{2m-2k}}{(2k)!! \sqrt{(2m-2k)!}} \right) + \sum_{m=0}^{\infty} (\beta \|B\| \, \|f\|)^{2m+1} \left(\sum_{k=0}^{m} \frac{(2b)^{2m+1-2k}}{(2k)!! \sqrt{(2m+1-2k)!}} \right) \right],$$

which converges since each summand in [...] is the Cauchy product of two converging power series. This proves the assertion.

Generalisation for $m>1$: Instead of $\sum_{v}$ in (4.3) with 2^n summands, we now get a sum with $(2m)^n$ similar summands. This implies the following estimations, which are analogous to (4.4) and (4.5)

$$\| Q_t^{(n)}(\boldsymbol{t}) \| \leq e^{t\|A\|}(2m\delta\gamma c_t)^n \sqrt{n!} \qquad \forall \boldsymbol{t} \in E_t^{(n)},$$

$$\| Q_{\beta/2}^{(n)}(\boldsymbol{t}) \|_{HS} \leq a\, d\, e^{\frac{\beta}{2}\|A\|}\, n!\,(2m\delta\gamma)^n \left(\sum_{k=0}^{[n/2]} \frac{(2b)^{n-2k}}{(2k)!!\sqrt{(n-2k)!}} \right) \qquad \forall \boldsymbol{t} \in E_{\beta/2}^{(n)}$$

for each $n \in \mathbb{N}$ with the constants $\delta = \max\{\|B_k\| \mid k=1,\ldots,n\}$ and $\gamma := \max\{\|f_k\| \mid k=1,\ldots,n\}$. □

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